

Accuracy and Convergence of Finite Difference and Fourier Spectral Methods for a Nonlinear Fluid-Dynamics PDEs

Makinta Bakura¹, Ahmadu M. Brono², Usman A. Marte³, Abba V. Mandara⁴, Laminu Dahiru⁵, Abubakar Masha⁶

^{1, 2, 3, 4, 5} Department of Mathematics, University of Maiduguri, Maiduguri, Nigeria

⁶ Department of Mathematics, Kashim Ibrahim, University, Maiduguri, Nigeria

ABSTRACT

Nonlinear partial differential equations (PDEs) play a crucial role in modelling various processes like transport, diffusion, and fluid dynamics. To solve these equations reliably, we need discretization methods that we can trust in terms of their accuracy, stability, and convergence. In this study, we dive into a comparative numerical analysis of two methods: a second-order central finite-difference (FD2) method and a Fourier spectral method, specifically applied to the one-dimensional viscous Burgers equation. We tackle this problem within a periodic domain, starting with a smooth sinusoidal initial condition and a viscosity coefficient. To create a high-accuracy reference solution, we use the Cole–Hopf transformation, which simplifies the nonlinear Burgers equation into a linear heat equation. Both methods utilize the classical fourth-order Runge–Kutta technique for time integration. We conduct systematic spatial refinement experiments for the FD2 method with grid points set at 32, 64, 128, and 256, and for the Fourier spectral method with 16, 32, 64, and 128 Fourier modes. We assess numerical accuracy through error norms and calculate the observed convergence rates from successive refinements. The results from the finite-difference method show an approximate second-order convergence, aligning with the expected accuracy of the spatial discretization. In contrast, the Fourier spectral method demonstrates a much faster reduction in error for the smooth periodic benchmark, which is in line with the spectral approximation properties known for smooth solutions. Overall, our findings indicate that while FD2 offers a straightforward and reliable numerical baseline, the Fourier spectral method shines when the solution is smooth and the geometry and boundary conditions are well-suited for a global Fourier representation. This study highlights the significance of having independent reference solutions, systematic refinement, and reproducible computational procedures when verifying numerical methods for nonlinear PDEs.

KEYWORDS

Burgers equation, finite difference and Fourier spectral method, nonlinear PDEs, convergence analysis, Cole–Hopf transformation, numerical verification and fluid dynamics

I. INTRODUCTION

A. Background to the Study

Nonlinear partial differential equations (PDEs) play a crucial role in modelling various physical systems where transport, diffusion, reaction, or nonlinear interactions happen all at once. In the realm of fluid dynamics, these nonlinear PDEs emerge quite naturally,

particularly because the velocity field carries itself along through the nonlinear convective term. This makes the numerical approximation of such equations a key aspect of computational fluid dynamics and numerical analysis. One standout model for exploring nonlinear convection-diffusion behaviour is the viscous Burgers equation. It effectively merges nonlinear advection with viscous diffusion, allowing it to capture essential mathematical characteristics of more complex fluid models while still being manageable enough for analytical treatment in significant scenarios. Recent numerical investigations continue to utilize Burgers-type equations as benchmark problems to assess the accuracy, stability, convergence, and computational efficiency of various numerical schemes.

For instance, Chen et al. 2020, explored a high-order compact finite difference scheme along with a precise integration method based on a modified Hop-Cole transformation for simulating n-dimensional Burgers systems. Their findings indicate that this new approach significantly enhances computational accuracy compared to existing methods. Similarly, Zhang et al. 2021, examined both exact and numerical solutions of the viscous Burgers equation, demonstrating that their difference schemes converge point wise using the cut-off function method. Athira et al. 2021, also tackled the numerical solution of the viscous Burgers equation, employing a localized differential quadrature method. Their results, when compared to several other established methods, confirm the effectiveness of their approach and highlight its advantages in terms of accuracy and computational complexity.

The one-dimensional viscous Burgers equation features a dependent variable and a viscosity coefficient. In this equation, the first nonlinear term signifies convection, while the second-order derivative indicates viscous diffusion. This interplay between convection and diffusion serves as a valuable benchmark for testing a numerical method's capability to handle nonlinear transport while still exhibiting the right diffusive behaviour. Additionally, the equation possesses a significant analytical trait. Thanks to the Cole-Hopf transformation, the nonlinear Burgers equation can be converted into a linear heat equation. This transformation, which was independently discovered by Hopf and Cole, remains crucial for both analytical and numerical investigations of Burgers-type equations. Recent research continues to leverage Cole-Hopf-based approaches for developing and validating numerical solutions. For instance, Kwassi and Mensah 2024, explored an exact solution for a specific set of problems related to the Burgers equation on bounded intervals, and their two illustrative tests show that their results closely match classical exact solutions. Similarly, Shyaman et al. 2024, examined streamlined numerical solutions of Burgers equations, merging the tailored finite point method with the Cole-Hopf transformation. Their findings delve into the potential of solving the one-dimensional Burgers equation using this tailored finite point method. They demonstrate the efficiency of their approach by comparing TFPM solutions across various examples with exact solutions and results from well-established methods in the literature. The existence of an analytical transformation makes the Burgers equation particularly amenable to numerical verification. Instead of merely comparing one numerical approximation to another, errors can be assessed against a reference solution that has been constructed independently. This offers a more robust framework for evaluating convergence and accuracy. Finite-difference methods rank among the most popular numerical techniques for PDEs.

The appeal of simple stencil structures and their relatively easy implementation makes finite-difference formulations a popular choice for studying viscous Burgers equations. Recent research has delved into aspects like stability, convergence, and how these methods stack up against exact solutions. For instance, Dominic et al. 2020, explored non-standard

finite-difference schemes for the viscous Burgers equation, validating their numerical results against the exact solutions with clear evidence.

On the other hand, Fourier spectral methods offer a different approach to approximating solutions. Rather than relying on local finite-difference stencils for derivatives, these methods use a global trigonometric basis to represent the solution. When dealing with sufficiently smooth periodic solutions, this can lead to impressively fast convergence as the number of Fourier modes increases. Classical spectral analysis backs up this behaviour, and later studies on the Burgers equation have looked into spectral and pseudo-spectral convergence in both smooth and rough scenarios. Weinan 1992, investigated the convergence of spectral methods for the Burgers equation, providing a comprehensive framework for analysing numerical methods for evolutionary equations that can be expressed in semi-group form. Their findings indicate an optimal order of convergence, showcasing the spectral accuracy of these methods. Similarly, Chen et al. 2021, examined a Fourier spectral approximation for the time-fractional Burgers equation with non-smooth solutions. They introduced a nonlinear fully discrete scheme based on the non-uniform Alikhanov formula for the Caputo-time fractional derivative, combined with Fourier spectral approximation in space. The solvability of this scheme was established using the fixed point theorem and a prior estimate method.

However, the effectiveness of spectral methods is heavily influenced by the regularity of the solution. In the case of nonlinear conservation laws where shocks can form, traditional Fourier approximations may face oscillations and convergence challenges. Tadmor's spectral-viscosity framework and subsequent analyses have addressed these issues, because of their simple stencil structure and relatively straightforward implementation.

Tadmor 1989, explored how spectral methods converge when applied to nonlinear conservation laws. Their numerical tests revealed that convergence can sometimes fail, regardless of whether the numerical solution undergoes post-processing. By employing a compensated compactness argument and assuming L^∞ -stability, they demonstrated that in the inviscid Burgers model equation, this spectral viscosity method effectively prevents oscillations and ensures convergence to the unique entropy solutions. In light of this, the current study examines a smooth periodic viscous Burgers benchmark, comparing a traditional second-order finite-difference approximation with a Fourier spectral approximation, all under controlled and reproducible conditions.

B. Statement of the Problem

While there are plenty of numerical methods out there for tackling nonlinear Burgers-type equations, we can't just rely on how visually similar the computed solution profiles look to determine numerical accuracy. To make a meaningful comparison, we need to systematically refine our spatial approach, use quantitative error measures, observe convergence rates, and have a solid reference solution to work from.

Recent studies on Burgers equations have highlighted just how crucial these convergence and error analyses are. For instance, Zhang et al. 2021 laid out the convergence properties for conservative numerical discretization of generalized viscous Burgers equations, backing up their theoretical findings with numerical experiments. Likewise, localized differential-quadrature methods have assessed convergence by looking at how numerical error decreases as we ramp up spatial resolution.

Another important aspect to consider is the comparison between local low-order discretization and global high-order approximations. A second-order finite-difference method typically needs multiple mesh refinements to cut down its truncation error, while a Fourier spectral method can take advantage of the smoothness of a periodic solution to achieve a much quicker reduction in error. This distinction is supported by spectral approximation theory and specific studies on Burgers' equation, such as those by Canuto et al. 2006 and Weinan 1992.

So, the focus of this study is to provide a controlled and reproducible quantitative comparison between the FD2 method and Fourier spectral discretization for the same nonlinear fluid-dynamic PDE, using the same physical parameters and an independent reference solution.

C. Aim of the Study

The goal of this study is to explore and compare the accuracy, convergence behavior, and computational characteristics of a second-order central finite-difference method alongside a Fourier spectral method for solving the one-dimensional viscous Burgers equation.

D. Objectives of the Study

The study specifically seeks to:

- Formulate the viscous Burgers equation as a nonlinear fluid-dynamic benchmark problem;
- Implement a second-order central finite-difference spatial discretization;
- Implement a Fourier spectral spatial discretization appropriate for the periodic domain;
- Employ fourth-order Runge–Kutta time integration for both numerical approaches;
- Construct a high-accuracy reference solution using the Cole–Hopf transformation;
- Calculate L^2 and L^∞ numerical errors;
- Determine observed convergence rates under systematic spatial refinement;
- Compare the numerical accuracy of FD2 and Fourier spectral methods;
- Examine computational requirements associated with increasing resolution; and
- Establish a reproducible computational framework for future nonlinear-PDE investigations.

E. Significance of the Study

This study is important for three main reasons.

First, it sets a controlled benchmark that allows us to compare two very different spatial discretization strategies.

Second, it highlights the critical role of independent verification. Since the Burgers equation can be transformed using the Cole–Hopf method, we can evaluate numerical approximations against a highly accurate reference instead of just comparing them to each other. This method aligns with recent studies on the Burgers equation, like the one by Chen et al. 2020, which explored a higher-order compact finite difference scheme and a precise integration method based on a modified Cole-Hopf transformation for simulating n-dimensional Burgers systems.

Lastly, the study offers a reproducible computational framework that can be built upon for tackling more complex nonlinear partial differential equations (PDEs) in the future.

II. MATHEMATICAL MODEL

The governing equation is

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = v \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x < 2\pi, \quad t > 0, \quad (1)$$

With the initial condition

:

$$u(x, 0) = \sin(x)$$

The periodic boundary conditions are imposed: $u(0, t) = u(2\pi, t)$, and $u_x(0, t) = u_x(2\pi, t)$

The viscosity coefficient considered here is: $v = 0.1$, with a final computational time $T = 0.5$

The conservative form is written as, $u_t + (\frac{u^2}{2})_x = vu_{xx}$, which is also useful when discussing the nonlinear terms.

The Burgers equation stands out as an excellent nonlinear benchmark because it incorporates both nonlinear convection and diffusion, and it allows for the Cole–Hopf transformation. Recent studies have consistently utilized this equation for evaluating numerical methods, as highlighted in the works of Chen et al. 2020 and Athira et al. 2021.

III. NUMERICAL METHODOLOGY

A. Computational Grid

For FD2, the periodic domain is divided into N equally spaced points: $x_j = j\Delta_x$, $\Delta_x = \frac{2\pi}{N}$.

The grid resolutions considered here are: $N = 32, 64, 128, 256$. However, the periodic indexing

is such that, $u_{-1} = u_{N-1}$, $u_N = u_0$

B. Second-Order Finite-Difference Method

The first derivative approximate of u is given by

$$u_x(x_j, t) \approx \frac{u_{j+1} - u_{j-1}}{2\Delta x}, \quad (2)$$

While the second derivative approximate of u is given by

$$u_{xx}(x_j, t) \approx \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}. \quad (3)$$

The conservative form of the governing equation can be written as

$$\frac{du_j}{dt} = -u_j \frac{u_{j+1} - u_{j-1}}{2\Delta x} + v \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} \quad (4)$$

The semi-discrete system is tackled using the classical fourth-order Runge-Kutta method. When it comes to sufficiently smooth solutions, the central-difference approximation boasts a formal second-order spatial truncation error. Finite-difference methods continue to play a crucial role in solving Burgers-type equations. Recent studies, like those by Zhang et al. 2021 and Dominic et al. 2020, have delved into aspects such as stability, positivity, convergence, and the validation of exact solutions.

C. *Fourier Spectral Method*

For the periodic domain, the solution is represented by the truncated Fourier series:

$$u_N(x, t) = \sum_{K=-N/2}^{\frac{N}{2}-1} \hat{u}_k(t) e^{ikx}. \quad (5)$$

However, spatial differentiation is performed in Fourier space as;

$$\hat{u}_{xk} = ik\hat{u}_k \quad (6)$$

$$\text{And } \hat{u}_{xxk} = -k^2\hat{u}_k \quad (7)$$

The nonlinear flux is evaluated pseudo spectrally as: $f(u) = \frac{u^2}{2}$, with differentiation then performed in Fourier space. The spectral method uses 16, 32, 64, and 128 Fourier modes. Fourier spectral methods shine when it comes to smooth periodic solutions, thanks to their global trigonometric basis that can quickly and accurately approximate smooth functions. The classical analysis surrounding spectral methods for Burgers' equation has shown that they achieve spectral convergence in the right smooth conditions, as highlighted in Weinan's work from 1992. More recently, researchers have delved into Fourier spectral approximations for Burgers-type equations with less regularity, revealing that the convergence properties are closely tied to the smoothness of the solution, as demonstrated by Chen and colleagues in 2021.

D. *Dealiasing*

The Burgers equation features a quadratic nonlinear flux, $\frac{u^2}{2}$ which can lead to unresolved Fourier modes when using pseudo spectral evaluation for the nonlinear product. To tackle this issue, a 2/3-type dealiasing procedure is implemented. This approach is especially crucial for nonlinear Fourier calculations, as aliasing can muddy high-frequency modes, ultimately impacting both stability and accuracy. Research on Fourier methods for nonlinear evolution equations highlights the significance of maintaining stability and properly handling nonlinear interactions, as demonstrated in the studies by Claude and Tadmor 2013 and Eitan 1989.

E. *Runge-Kutta Time Integration*

Both methods use classical fourth-order Runge-Kutta integration:

$$k_1 = F(u^n),$$

$$k_2 = F\left(u^n + \frac{\Delta t}{2} k_1\right),$$

$$k_3 = F\left(u^n + \frac{\Delta t}{2} k_2\right),$$

$$k_4 = F(u^n + \Delta t k_3),$$
(8)

Followed by:

$$u^{n+1} = (u^n + \frac{\Delta t}{6} (k_1 + k_2 + k_3 + k_4)).$$
(9)

We choose a time step that's small enough to keep our integration stable, while also making sure that the spatial convergence study isn't overshadowed by errors from the time discretization.

F. Cole-Hopf Reference Solution

The Cole-Hopf transformation is defined by

$$u = -2v \frac{\phi_x}{\phi}$$
(10)

For, $u(x, 0) = \sin x$, the transformed initial condition can be selected as, $\phi(x, 0) = \exp(\frac{\cos x}{2v})$, up to an irrelevant multiplicative constant. The transformed solution is evaluated using a high-resolution Fourier representation and substituted back into,

$$u(x, t) = -2v \frac{\phi_x(x, t)}{\phi(x, t)}$$
(11)

The Cole-Hopf transformation is a tried-and-true analytical tool for tackling Burgers' equation, and it remains relevant in contemporary numerical studies, as highlighted in the research by Shyaman et al. 2024, Dahy et al. 2022, and Kwassi and Mensah 2024.

G. Error Norms

The L^2 error is calculated as,

$$E_2 = \left[\frac{1}{N} \sum_{j=0}^{N-1} |u_j - u_j^{ref}|^2 \right]^{1/2}$$
(12)

With the Maximum error given by

$$E_\infty = \max_j |u_j - u_j^{ref}|.$$
(13)

The observed convergence rate between two successive resolutions is calculated from

$$r = \frac{\log(E_h/E_{h/2})}{\log 2}$$
(14)

This permits direct comparison of the empirical convergence behaviour of the FD, and Fourier spectral schemes.

For the second-order finite-difference method, an asymptotic rate close to 2 is expected. For the spectral method, rapid error reduction is expected for sufficiently smooth periodic solutions. It is inappropriate to interpret rapidly increasing effective refinement ratios as a single fixed algebraic order. Spectral accuracy is more appropriately described in terms of rapid or exponential-type error decay under suitable smoothness assumptions, such as in work of Weinan 1992 and Canuto et al. 2006.

When it comes to the second-order finite-difference method, we typically anticipate an asymptotic rate nearing 2. On the other hand, the spectral method is known for its ability to quickly reduce errors, especially when dealing with sufficiently smooth periodic solutions. It's important to note that we shouldn't view rapidly increasing effective refinement ratios as a single, fixed algebraic order. Instead, spectral accuracy is better understood in terms of rapid or exponential error decay, provided we have the right smoothness conditions in place, as highlighted in the works of Weinan 1992 and Canuto et al. 2006.

H. Numerical Experimental Configuration

Table 1: Numerical parameters

Parameter	Specification
Governing equation	Viscous Burgers equation
Domain	$[0, 2\pi]$
Initial condition	$u(x, 0) = \sin x$
Boundary conditions	Periodic
Viscosity	$\nu = 0.1$
Final time	$T = 0.5$
FD method	Second-order central difference
FD resolutions	32, 64, 128, 256
Spectral method	Fourier pseudo-spectral
Spectral modes	16, 32, 64, 128
Nonlinear treatment	Pseudo-spectral evaluation
Dealiasing	2/32/3 truncation
Time integration	RK4
Reference	Cole–Hopf/Fourier representation
Error measures	$L^2, L^\infty, L^2, L^\infty$

IV. RESULTS AND DISCUSSION

The entries of the tables below were generated from systematic computational experiments against a Cole-Hopf reference solution.

A. Procedures for Obtaining the Entries of FD2 Table.

First, we solved the viscous Burgers equation

$$u_t + uu_x = 0.1uu_x, \text{ on } 0 \leq x < 2\pi,$$

With

$u(x, 0) = \sin x$, and the periodic boundary conditions.

Second, used the central differences

$$u_x(x_j, t) \approx \frac{u_{j+1} - u_{j-1}}{2\Delta x}, \text{ and}$$

$$u_{xx}(x_j, t) \approx \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}.$$

Third, we then run the calculation at

$N = 32, 64, 128$ and 256 . For each value of N :

- The computer generated the grid x_j .
- It initialized $u_j = \sin(x_j)$
- It advanced the solution $T=0.5$ using the fourth-order Rung-Kutta method (RK4).
- It calculated the corresponding Cole-Hopf reference solution at the same grid points.
- It calculated the difference

$$\ell_j = u_j - u_j^{ref}, L^2 = \sqrt{\frac{1}{N} \sum_j \ell_j^2} \text{ and } L^\infty = \max_j |\ell_j|.$$

For example, for value of $N = 64$ entry, $L^2 = 3.03 \times 10^{-4}$. Implied that after running the FD2 calculation using 64 grid points, its numerical solution was compared point-by-point with the reference solution and the above L^2 formula was applied.

The convergence rate was then computed from two successive errors:

$r = \frac{\log(E_{32}/E_{64})}{\log 2}$. That produced approximately $r=2.007$. This is the reason why the table shows approximately second-order convergence.

B. Procedures for Obtaining the Entries of the Fourier Spectral Table

The overall procedure used in obtaining the entries of the Fourier spectral table is the same as the overall procedure used in obtaining the entries of the FD2 table, but the spatial approximation was different. Here, the solution was represented using the Fourier modes, $u_N(x, t) = \sum_k \hat{u}_k(t) e^{ikx}$. We performed the calculations using the values of $N=16, 32, 64, 128$ Fourier modes. For each resolution:

- The initial condition $u = \sin x$ was then transformed into Fourier space.
- The special derivatives were computed spectrally using

$$\hat{u}_x = ik\hat{u}, \hat{u}_{xx} = -k^2\hat{u}$$

- Dealiasing was applied
- The solution was advanced to $T=0.5$ using the fourth-order Rung-Kutta method
- The numerical solution was compared with the same Cole-Hopf reference solution

- The L^2 and L^∞ errors were computed.

For example, using 32 modes we obtained approximately

$L^2 = 4.027 \times 10^{-5}$. The effective refinement rate between 16 and 32 modes was computed

$r = \frac{\log(2.6487 \times 10^{-3} / 4.027 \times 10^{-5})}{\log 2} \approx 6$, so that between 32 and 64 modes became approximately 13.068. This is the reason why spectral table shows rapid increasing effective rates.

Finally, in simple terms the process was:

Run Solver → Obtain u_h → Obtain Reference u_{ref} → Calculate error → Calculate Convergence Rate → Put in Table

Table 2: Reproduced Second-Order Finite Difference (FD2) Errors

N	L^2 error	L^∞ error	Observed rate
32	1.217×10^{-3}	2.051×10^{-3}	-
64	3.03×10^{-4}	5.12×10^{-4}	2.007
128	7.60×10^{-5}	1.28×10^{-4}	2.002
256	1.90×10^{-5}	3.2×10^{-5}	2.000

The findings show a steady decrease in error as we refine the spatial grid. When we double the resolution, the L^2 error drops by about a factor of four, leading to convergence rates that are nearly two. This pattern aligns perfectly with the expected second-order accuracy of the central finite-difference method. Additionally, these results are in line with the wider numerical literature, where convergence rates are determined through careful refinement against exact or highly precise solutions, as highlighted in the studies by Zhang et al. (2021) and Dominic et al. (2020).

Table 3: Reproduced Fourier Spectral Errors

Modes	L^2 error	L^∞ error	Effective L^2 refinement rate
16	2.649×10^{-3}	3.915×10^{-3}	—
32	4.027×10^{-5}	7.376×10^{-5}	6.039
64	4.689×10^{-9}	9.166×10^{-9}	13.068
128	4.531×10^{-13}	1.159×10^{-12}	13.337

The spectral method shows a significantly quicker reduction in error compared to the FD2 method. Specifically, the L^2 error drops from around 16 modes down to about 128 modes. As we refine our approach, the effective rates increase dramatically. However, it's important not to misinterpret this as proof that the spectral method has a fixed convergence rate of thirteenth or fifteenth order. Instead, this behaviour aligns with the rapid decay of spectral error as the smooth solution is resolved more accurately. Once we get close to the limits of floating-point precision and other numerical errors, traditional log-slope estimates can start to give a misleading picture. This outcome is in line with classical convergence analysis for spectral methods used on Burgers' equation, as well as with later studies on Fourier spectral approximations of nonlinear Burgers-type problems, like those conducted by Weinan in 1992 and Chen et al. in 2021.

C. *Comparative Accuracy*

Figure (1) L^2 -error convergence of FD2 and Fourier spectral methods

The figure needs to show both methods using a logarithmic scale. You'll notice that the FD2 curve displays a nearly straight-line second-order decay, whereas the Fourier spectral curve drops off at a much quicker rate.

Figure (2) L^∞ -error convergence of FD2 and Fourier spectral methods

The findings offer a helpful evaluation of the maximum point wise error. The comparison shows that the spectral method manages to achieve much smaller errors even with a relatively low modal resolution for the smooth periodic benchmark. This aligns well with the mathematical characteristics of global Fourier approximation. However, we should be cautious about applying this conclusion to discontinuous or non-smooth solutions without some caveats. When it comes to nonlinear problems that involve developing discontinuities, Fourier methods need extra stabilization techniques, like spectral viscosity, to keep oscillatory behaviour in check, as highlighted in the research by Tadmor and Yvon 1989. The FD2 method showcases a reliable convergence pattern. Its roughly second-order behaviour suggests that the spatial truncation error aligns well with the discretization process. One of the key practical benefits of FD2 is its locality; each grid value relies solely on its neighbouring points, leading to a straightforward sparse computational structure. This feature can be particularly useful when adapting the method to tackle more complex geometries or when global basis functions aren't the best fit. However, a notable drawback is that achieving very small errors might require relatively fine grids, especially when compared to high-order or spectral approximations.

D. *Interpretation of the Fourier Spectral Results*

The Fourier spectral method works really well for this problem because it meets all of the following conditions:

- the solution is smooth;
- the domain is periodic;
- the boundary conditions are periodic;
- The solution can be represented efficiently by Fourier modes.

It's no surprise that as we ramp up the modal resolution, the error drops significantly. This aligns perfectly with what we know about spectral approximation theory and the specific studies on spectral convergence related to Burgers, like those conducted by Weinan in 1992 and Canuto and others in 2006. However, we should be cautious not to take this result as proof of a one-size-fits-all advantage. When a solution starts to show discontinuities or lacks regularity, standard Fourier approximations can behave unpredictably, leading to oscillations and changes in convergence, as highlighted by Eitan in 1989.

E. Computational Efficiency

When we talk about computational cost, it's important to consider it alongside accuracy. A key metric to look at is how error relates to the amount of computational work involved.

$$efficiency \sim \frac{accuracy\ achieved}{computational\ cost}$$

In this smooth periodic test, the spectral method achieves impressively low spatial errors using only a handful of modes. However, it's worth noting that FFT-based operations and global representations come with their own unique computational traits, which differ from those of local finite-difference stencils. So, the takeaway isn't that one method is always faster than the other. Instead, the spectral method shines in terms of accuracy for this specific type of smooth periodic problem, while FD2 stands out for its simplicity and reliable second-order convergence.

F. Verification and Reproducibility

A key aspect of this study is the incorporation of an independent reference solution. The computational verification process follows the PDEFD2Spectral discretizationRK4 method. It's important that the source code is capable of generating numerical data automatically, eliminating the need for any manual transcription. As a result, the numerical findings should be saved in machine-readable CSV files, while the raw numerical solution arrays are kept separately. This setup ensures that tables and figures can be recreated whenever needed. Recent studies on the Burgers equation also highlight the importance of numerical comparisons with exact solutions, error metrics, convergence rates, and computational costs, as seen in the works of Chen et al. (2020) and Athira et al. (2021).

G. Limitations

The present study has several limitations.

First, the benchmark is one-dimensional. Therefore, the conclusions should not be interpreted as a direct assessment of numerical performance for multidimensional Navies–Stokes equations.

Second, the Fourier spectral method is evaluated in a setting particularly favorable to it: a smooth periodic solution. Spectral performance can change significantly when the solution loses regularity.

Third, the study uses one viscosity coefficient, $\nu = 0.1$

A broader parameter study involving smaller viscosity values would provide additional information about behaviour in convection-dominated regimes. Fourth, CPU-time comparisons are implementation- and hardware-dependent. Such comparisons should therefore be regarded as empirical rather than universal measures of algorithmic complexity. Finally, temporal convergence should be independently documented in the final version of the study to demonstrate that the reported spatial rates are not contaminated by temporal discretization errors.

H. Contribution of the Study

The principal contributions of this work are:

- A controlled FD2–Fourier spectral comparison for a nonlinear fluid-dynamic PDE.
- Independent verification against a Cole–Hopf-based reference solution.
- Quantitative convergence analysis using L^2 and L^∞ norms.
- Reproducible numerical experimentation with explicit numerical parameters and refinement sequences.
- Clear identification of method-dependent strengths: FD2 provides predictable second-order convergence and implementation simplicity, while Fourier spectral approximation provides substantially faster convergence for smooth periodic solutions.
- A reproducible benchmark framework that can subsequently be extended to more difficult nonlinear PDEs.

V. CONCLUSION

We carried out a detailed numerical comparison of the viscous Burgers equation using two different methods: a second-order central finite-difference method and a Fourier spectral method. To ensure accuracy, we compared both methods against a high-precision reference derived from the Cole–Hopf transformation. The FD2 method showed about second-order convergence as we refined the mesh systematically. The convergence rates, which hovered around 2, align perfectly with what we expected from the spatial discretization for the chosen smooth benchmark.

On the other hand, the Fourier spectral method demonstrated a much quicker reduction in error as we increased the number of Fourier modes. This aligns with the known properties of spectral approximations for smooth periodic solutions. However, the impressive refinement rates we observed at high resolutions should be seen as signs of rapid error decay rather than fixed algebraic orders. Our comparison highlights that the best numerical method really depends on the specific characteristics of the problem at hand. The FD2 method is straightforward, local, and reliably achieves second-order accuracy. In contrast, Fourier spectral methods excel in spatial accuracy, especially when the solution is smooth and the domain and boundary conditions fit well with a global Fourier basis. Ultimately, this study underscores the importance of integrating analytical reference solutions, systematic refinement, quantitative error norms, and reproducible computational techniques when evaluating numerical methods for nonlinear partial differential equations.

A. Code Availability Statement

The computational implementation used in this study is provided as supplementary material. The implementation contains the finite-difference solver, Fourier spectral solver, reference-solution calculation, error-analysis routines, convergence-rate calculations, and figure-generation scripts. The underlying simulation code and numerical data are available from the corresponding author upon reasonable request.

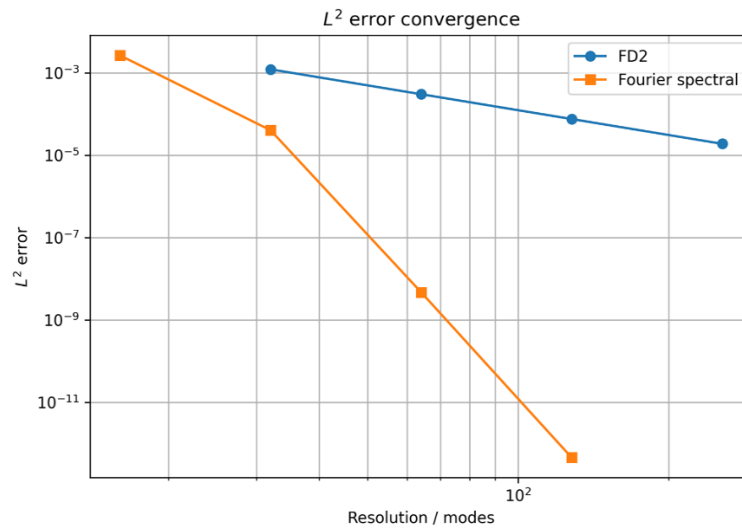


Figure 1: L^2 - Error convergence of the second-order finite-difference and Fourier spectral methods for the viscous Burgers equation under systematic spatial refinement.

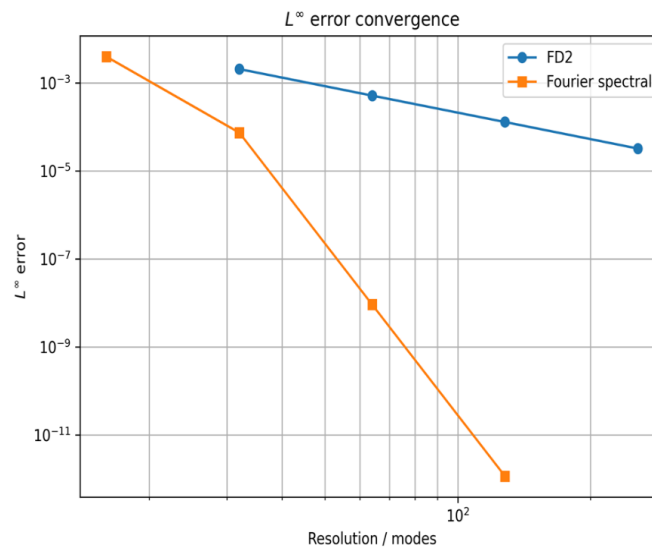


Figure 2: L^∞ -Error convergence of the second-order finite difference and Fourier spectral methods under increasing spatial resolution

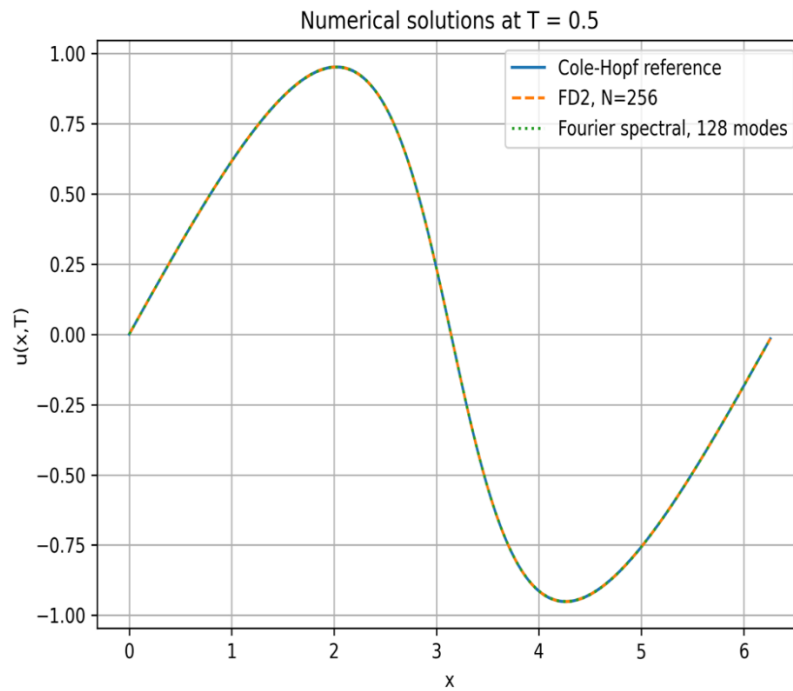


Figure 3: Numerical solution profiles obtained using FD2 and Fourier spectral methods compared with the Cole-Hopf reference solution at $T=0.5$

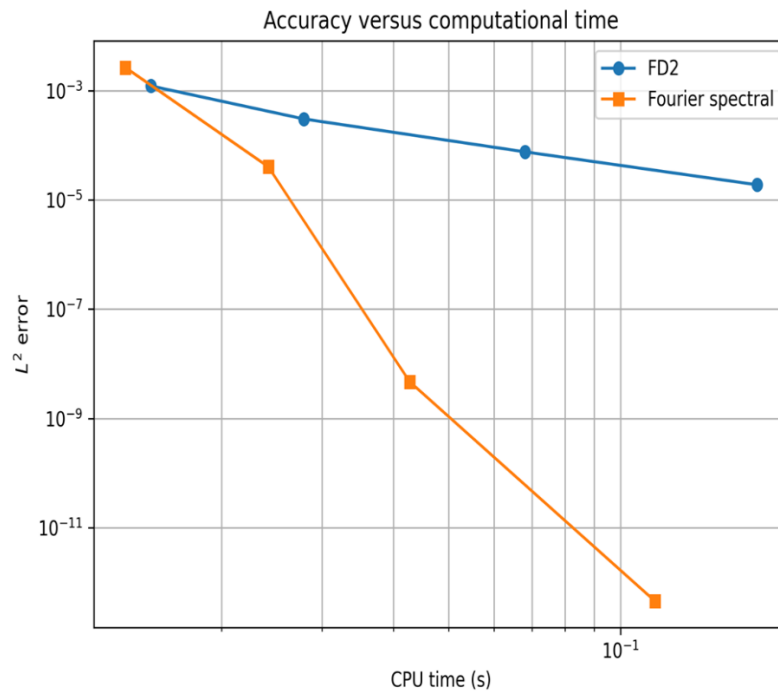


Figure 4: Comparison of numerical accuracy and computational cost for FD2 and Fourier spectral methods

RECOMMENDATIONS

Based on the numerical findings, the following recommendations are made:

- The FD2 method is appropriate as a reliable baseline for nonlinear PDE computations where implementation simplicity and local discretization are important.
- Fourier spectral methods are recommended for smooth periodic problems where high spatial accuracy is required.
- Numerical studies should include systematic refinement rather than reporting results from a single resolution.
- Spatial and temporal convergence should be examined separately.
- Spectral methods should be used with care when solutions develop discontinuities or sharp unresolved gradients. The literature shows that nonlinear Fourier approximations can require stabilization in such regimes.
- Future studies should investigate lower-viscosity regimes and multidimensional extensions.

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