The Gbenga Gideon Integral Transform and Its Application to Third-Order Ordinary Differential Equations

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ABSTRACT

This study presents a four-parameter generalization of the Laplace transform, here called the Gbenga Gideon (G.G.) integral transform (GGIT), with kernel $pv^m\{e^{-qv^w t}\}$, and develops its operational calculus for the solution of third-order ordinary differential equations. Elementary transform pairs are derived for constant, power, exponential, trigonometric and hyperbolic functions, and the fundamental operational properties — linearity, first and second shifting, change of scale, differentiation, multiplication by the independent variable, Laplace–GGIT duality and convolution — are stated and proved. A compact formula for the transform of the n-th derivative is established, with the third-order specialization required for third-order initial-value problems singled out. The parameter choices through which the G.G. kernel reduces to thirty-eight named transforms from the recent literature, including the Laplace, Sumudu, Aboodh, Elzaki, Sawi, Shehu and Sadik transforms, are tabulated. The method is then applied to third-order constant-coefficient equations with trigonometric and exponential-polynomial forcing terms. The GG Integrals Transform solutions were bench-marked against key existing transforms (Laplace, Aboodh, and Elzaki) and the results were shown to agree.

KEYWORDS

G.G. integral transform, third-order ordinary differential equation, integral transform, Laplace transform, Aboodh transform, Elzaki transform, analytical solution

I. INTRODUCTION

Integral transforms provide a systematic means of converting differential and integral equations into algebraic relations involving an auxiliary variable. For an appropriate kernel, differentiation with respect to the original variable is transformed into multiplication by a transform parameter, together with terms determined by the initial conditions. The resulting algebraic equation can then be solved using factorization, partial fractions, transform tables, or convolution, after which the inverse transform is applied.

Although the Laplace transform remains the most widely used transform in this class, substantial recent attention has been devoted to relate Laplace-type transforms involving alternative powers and normalizations. These include the Sumudu, Aboodh, Mohand, Elzaki, Sadik, Sawi, Shehu, Natural, and Jafari transforms (Watugala, 1993; El-Mesady et al., 2021;

Kuffi, 2023; Jafari, 2021). Many of these transforms can be represented as particular cases of the general four-parameter kernel [pv^m e^{-qv^{wt}.] Introducing this unified kernel, together with its associated operational calculus, provides a concise framework that avoids repeating the same derivations for each individual transform.

Third-order differential equations and their treatment in the transform domain have also received considerable attention. The structural and geometric properties of third-order ordinary and partial differential operators have been examined by Godlinski (2008), Chandrasekar et al. (2005), Berkovich (2001), Léon and Scárdua (2021), Wang and Xu (2024), and Sousa et al. (2024). In addition, direct-series, operational-matrix, and elastic-transform techniques for solving third-order equations have been developed by Hashim and Alshbool (2019), Dong et al. (2023), and Sattaso et al. (2019), while numerical comparisons of solution methods have been reported by Aghayeva et al. (2024) and Crouch and Grossman (1993). Broader discussions of integral transforms and their engineering applications are presented by Cotta (2020), Saitoh (2020), Wolf (2013), Wazwaz (2015), Banerjea and Mandal (2023), Atangana and Akgül (2023), Abels (2011), Jeffrey and Dai (2008), Burton (2005), Srivastava (2020), and Chicone (2006). Transform-based approximation methods for higher-order equations are further discussed by Ramadan et al. (2017), Saadeh et al. (2023), and Gautschi (1961), while asymptotic methods for integrals are surveyed by Wong (2001).

This article develops a systematic treatment of a four-parameter transform, termed the Gbenga–Gideon (GG) integral transform, and employs it to analyse third-order constant-coefficient ordinary differential equations, which arise, for instance, in thin-film flow (Momoniat, 2009) and in numerous boundary-value problems involving third-order linear operators (Greguš, 2012; Pelloni, 2007). It offers four main contributions. First, it provides a rigorous definition of the GG Integral transform and establishes a sufficient condition ensuring its existence. Second, it assembles the basic transform pairs and operational rules required for third-order applications, including the transform of the n-th derivative. Third, it lists the parameter configurations under which the GG kernel reproduces thirty-eight transforms reported in recent studies. Fourth, it applies the transform to third-order equations subject to trigonometric and exponential-polynomial forcing, and verifies the resulting solutions by comparison with those obtained from the Laplace, Aboodh and Elzaki methods.

II. DEFINITION AND EXISTENCE OF THE G.G. INTEGRAL TRANSFORM

Let f (t) be a real- or complex-valued function defined fort ≥ 0. For fixed parameters p, q, m and w, the G.G. integral transform off is defined by

$$G[f(t)](v) \;=\; G(v) \;=\; \int_0^\infty pv^m \, e^{-qv^{w}t} \, f(t)\, dt, \tag{1}$$

Provided the integral converges. Whenever this integral exists it is denoted G {f (t)} (v) = G (v). In the real-valued setting one takes p > 0, q > 0 and v > 0, and requires q v^w to lie to the right of the exponential-growth abscissa off, in direct analogy with the Laplace transform.

A. Definition 1 (exponential order)

A function f is of exponential order α if there exist constants M > 0 and T ≥ 0 such that

$[f(t)] \leq M\, e^{\alpha t},\ t \geq T.$

B. Theorem 1 (existence of the G.G.integral transform)

If f is piecewise continuous on every finite subinterval of [0, ∞) and is of exponential order α, then G {f (t)} (v) exists for every v such that

$$qv^{w} > \alpha. \tag{2}$$

C. Proof

Choose T so that the inequality in Definition 1 holds, and split the defining integral at T:

$$G(v) = pv^{m}\int_{0}^{T} e^{-qv^{w}t} f(t)\, dt + pv^{m}\int_{T}^{\infty} e^{-qv^{w}t} f(t)\, dt$$

The first integral exists because its integrand is piecewise continuous on a finite interval. For the second integral,

$$\left[pv^{m}e^{-qv^{w}t}f(t)\right] \leq Mpv^{m}e^{-(qv^{w}-\alpha)t},\ t \geq T.$$

Since q v^w > α, the right-hand side is integrable on [T, ∞):

$$\int_{T}^{\infty} Mpv^{m}e^{-(qv^{w}-\alpha)t}\, dt = \frac{Mpv^{m}e^{-(qv^{w}-\alpha)T}}{qv^{w} - \alpha} < \infty.$$

By the comparison test the second integral converges absolutely, and hence G {f (t)} (v) exists.

D. Definition 2 (inverse G.G. integral transform)

If G (v) = G {f (t)} (v) and the inverse operation exist on the relevant domain, then f is called the inverse G.G. transform of G, written

$$f(t) = G^{-1}[G(v)](t). \tag{3}$$

In the applications of Section 10, inversion is carried out through the elementary transform pairs of Section 3 together with the specialisation of the G.G. transform to the Laplace, Aboodh and Elzaki transforms.

III. ELEMENTARY G.G. TRANSFORM PAIRS

Table 1 lists the elementary transform pairs used throughout the paper. Representative proofs follow the table; the remaining entries are obtained in the same way.

f(t)	**G{f(t)}(v)**	**Condition**
a	$\frac{apv^m}{qv^w}$	$qv^w > 0$
t^n	$\frac{n!\ pv^m}{(qv^w)^{n+1}}$	$n = 0,1,2,\dots$
t^a	$\frac{\Gamma(\alpha+1)\, pv^m}{(qv^w)^{\alpha+1}}$	$\alpha > -1$
e^{kt}	$\frac{pv^m}{qv^w - k}$	$qv^w > k$
$\sin kt$	$\frac{kpv^m}{q^2v^{2w} + k^2}$	$k \in \mathbb{R}$
$\cos kt$	$\frac{pqv^{m+w}}{q^2v^{2w} + k^2}$	$k \in \mathbb{R}$
$\sinh kt$	$\frac{kpv^m}{q^2v^{2w} - k^2}$	$qv^w > k$
$\cosh kt$	$\frac{pqv^{m+w}}{q^2v^{2w} - k^2}$	$qv^w > k$

A. Representative Proofs

1. Constant. With f (t) = a t^0,

$$G[a] = a\,pv^m \int_0^\infty e^{-qv^w t}\,dt = \frac{apv^m}{qv^w}$$

Using the substitution x = q v^w t.

2. Power function. With f (t) = t^n and x = q v^w t,

$$G[t^n] = pv^m \int_0^\infty \left(\frac{x}{qv^w}\right)^n e^{-x}\,\frac{dx}{qv^w} = \frac{n!\ pv^m}{(qv^w)^{n+1}}$$

Since ∫₀^∞ x^n e^ {−x} dx = Γ (n+1) = n! For n = 0, 1, 2,

3. Exponential function. Directly,

$$G[e^{kt}] = pv^m \int_0^{\infty} e^{-(qv^w-k)t}\, dt = \frac{pv^m}{qv^w - k},\ qv^w > k.$$

4. Sine and cosine. Writing sin kt = (e^ {ikt} − e^ {−ikt})/2i and cos kt = (e^ {ikt} + e^ {−ikt})/2, and applying the exponential pair with k replaced by ±ik,

$$G[sin\, kt] = \frac{1}{2i}\left(\frac{pv^m}{qv^w - ik} - \frac{pv^m}{qv^w + ik}\right) = \frac{kpv^m}{q^2v^{2w} + k^2}$$

And the cosine pair follows in the same way with the plus sign. The hyperbolic pairs (e) follow identically from sinh kt = (e^ {kt} − e^ {−kt})/2 and cosh kt = (e^ {kt} + e^ {−kt})/2, replacing k by ±k rather than ±ik, which changes + k² in the denominator to − k².

IV. FUNDAMENTAL OPERATIONAL PROPERTIES

A. Linearity

If G {f_1 (t)} (v) = G_1 (v) and G {f_2 (t)} (v) = G_2 (v), then for arbitrary constants a and b,

$$G[af_1(t) + bf_2(t)](v) = aG_1(v) + bG_2(v), \tag{4}$$

Which follows directly from the linearity of the defining integral.

B. First shifting theorem

If G {f (t)} (v) = G (v), then for real k,

$$G[e^{kt}f(t)](v) = pv^m \int_0^{\infty} e^{-(qv^w-k)t}\, f(t)\, dt = G(qv^w - k), \tag{5}$$

So multiplication off (t) by e^ {kt} shifts the parameter q v^w to q v^w − k inside G. Combined with Table 1, this gives

$$G[e^{kt}t^n](v) = \frac{n!\ pv^m}{(qv^w - k)^{n+1}}$$

$$G[e^{kt}sin\, bt](v) = \frac{bpv^m}{(qv^w - k)^2 + b^2},\quad G[e^{kt}cos\, bt](v) = \frac{pv^m(qv^w - k)}{(qv^w - k)^2 + b^2}$$

And, since the derivation replaces ±ib by ±b throughout,

$$G[e^{kt}sinh\, bt](v) = \frac{bpv^m}{(qv^w - k)^2 - b^2}, \quad G[e^{kt}cosh\, bt](v) = \frac{pv^m(qv^w - k)}{(qv^w - k)^2 - b^2}$$

(In the source thesis these last two pairs are labelled "sin bt" / "cos bt", but their own proofs expand e^ {bt} ± e^ {−bt} over 2 — the hyperbolic definitions — and the denominators carry $-b^2$, the hyperbolic signature; the labels sinh bt / cosh bt used here match the working.)

C. Second shifting theorem

Let H denote the Heaviside unit step function and set f_a (t) = H (t − a) f (t − a) for a > 0. If G {f (t)} (v) = G (v), then

$$G[H(t-a)\, f(t-a)](v) = e^{-aqv^w}\, G(v). \quad (6)$$

Indeed, splitting the integral at t = a and substituting u = t − a in the surviving piece,

$$G[H(t-a)f(t-a)](v) = pv^m \int_a^\infty e^{-qv^w t} f(t-a)\, dt = pv^m e^{-aqv^w} \int_0^\infty e^{-qv^w u} f(u)\, du = e^{-aqv^w} G(v).$$

D. Change of scale

If G {f (t)} (v) = G (v), then for a > 0,

$$G[f(at)](v) = \frac{1}{a} G\left(\frac{v}{a}\right), \quad (7)$$

Obtained from the substitution u = at, which sends q v^w t to q (v/a) ^w u and dt to du/a.

V. TRANSFORM OF DERIVATIVES

Assume f and its first n − 1 derivatives are continuous and of exponential order, f^ (n) is piecewise continuous, and the boundary terms at infinity vanish. Integration by parts gives the first-derivative formula

$$G[f'(t)](v) = qv^w G(v) - pv^m f(0). \quad (8)$$

Proof. Integrating by parts with dz = e^ {−qv^w t} dt,

$$G[f'(t)](v) = pv^m\left[\left(e^{-qv^w t} f(t)\right)\right] \Big|_0^\infty + qv^w pv^m \int_0^\infty e^{-qv^w t}\, f(t)\, dt = -pv^m f(0) + qv^w G(v).$$

Applying the identity recursively (replace f by f′, then by f″) yields, for the second and third derivatives,

$$G[f''(t)](v) = q^2 v^{2w} G(v) - pqv^{m+w} f(0) - pv^m f'(0), \quad (9)$$

$$G[f'''(t)](v) = q^3 v^{3w} G(v) - pq^2 v^{m+2w} f(0) - pqv^{m+w} f'(0) - pv^m f''(0). \quad (10)$$

Equation (10) is the identity used throughout Section 10 for third-order equations. The next member of the sequence, needed nowhere else in the paper but recorded for completeness, is

$$G\left[f^{(4)}(t)\right](v) = q^4 v^{4w} G(v) - pq^3 v^{m+3w} f(0) - pq^2 v^{m+2w} f'(0) - pqv^{m+w} f''(0) - pv^m f'''(0). \quad (11)$$

A. Theorem 2 (general derivative formula)

For every integer n ≥ 1,

$$G\left[f^{(n)}(t)\right](v) = (qv^w)^n G(v) - \sum_{j=0}^{n-1} pq^{n-1-j} v^{m+(n-1-j)w} f^{(j)}(0) \quad (12)$$

Proof. The case n = 1 is (8). Assume (12) holds for n. Applying the first-derivative identity to f^ (n) gives

$$G\left[f^{(n+1)}(t)\right](v) = qv^w G\left[f^{(n)}(t)\right](v) - pv^m f^{(n)}(0).$$

Substituting the induction hypothesis for G {f^ (n) (t)} (v) and absorbing q v^w into the sum re-indexes it from j = 0 ton, which is (12) with n replaced by n + 1.

VI. MULTIPLICATION BY THE INDEPENDENT VARIABLE

Differentiating the defining integral with respect to the parameter q v^w brings down a factor of −t under the integral sign, so that, for every integer n ≥ 1,

$$G[t^n f(t)](v) = (-1)^n \frac{\partial^n G(v)}{(\partial(qv^w))^n} \quad (13)$$

In particular, the first-order case is

$$G[t\, f(t)](v) = -\frac{\partial G(v)}{\partial(qv^w)}$$

And the identities for t^2 f (t), t^3 f (t), and higher powers follow by repeated differentiation. This property is a standard companion to the derivative formulas of Section 5 and is not required for the third-order applications of Section 10.

VII. DUALITY AND CONVOLUTION

A. *Theorem 3 (Laplace–G G duality)*

Let $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ be the (ordinary) Laplace transform off. Comparing this with the defining integral (1) at $s = q\,v^w$ gives

$$G(v) = pv^m F(qv^w). \tag{14}$$

Equation (14) shows that the G.G. transform is the ordinary Laplace transform evaluated at $s = q\,v^w$ and rescaled by $p\,v^m$; it is the reason the G.G. transform inherits the Laplace transform's operational calculus while allowing the four-parameter re-normalization recorded in Section 9.

B. *Theorem 4 (convolution theorem)*

Let $(f_1 * f_2)(t) = \int_0^t f_1(x)\, f_2(t - x)\,dx$. If $G_1(v) = G\{f_1(t)\}(v)$ and $G_2(v) = G\{f_2(t)\}(v)$, then

$$G\left[\left(\int_0^t f_1(x) f_2(t-x)\,dx\right)\right](v) = \frac{G_1(v)\,G_2(v)}{pv^m}, \tag{15}$$

Equivalently $f_1 * f_2 = G^{-1}\{G_1G_2/(pv^m)\}$. The factor $1/(p\,v^m)$ is required for consistency with the duality relation (14): when $p = 1$ and $m = 0$ (the Laplace case) it disappears and (15) reduces to the classical Laplace convolution theorem.

Proof. By definition,

$$G\left[\left(\int_0^t f_1(x) f_2(t-x)\,dx\right)\right](v) = pv^m \int_0^\infty \int_0^t f_1(x) f_2(t-x)\,dx\, e^{-qv^w t}\,dt$$

The double integral is taken over the region $0 \le x \le t < \infty$. Reversing the order of integration and substituting $z = t - x$ in the inner integral gives

$$= pv^m \int_0^\infty f_1(x) e^{-qv^w x} \int_0^\infty f_2(z) e^{-qv^w z}\,dz\,dx$$

$$= G_2(v) \int_0^\infty f_1(x) e^{-qv^w x}\,dx = \frac{G_1(v)\,G_2(v)}{pv^m},$$

Since the surviving integral equals $G_1(v)/(p\,v^m)$ by definition (1).

VIII. SELECTED BESSEL-FUNCTION TRANSFORM PAIRS

For completeness, the source thesis also records the G.G. transforms of the Bessel functions J_0 and J_1 of the first kind. In the parameter convention of Section 2,

$$G[J_0(t)](v) = \frac{pv^m}{\sqrt{(qv^w)^2+1}}, \quad G[J_0(kt)](v) = \frac{pv^m k^w}{\sqrt{(qv^w)^2+k^{2w}}} \tag{16}$$

$$G[J_1(t)](v) = \frac{pv^m\left(\sqrt{(qv^w)^2+1}-qv^w\right)}{\sqrt{(qv^w)^2+1}}, \tag{17}$$

With the scaled pair G {J_1 (kt)} (v) obtained by replacing 1 with k^{2w} under the square root and dividing by k^m. These entries are retained as transform-table data from the source thesis; using them in an inversion requires the corresponding Bessel-function normalization and transform domain, which lies outside the third-order applications of Section 10.

IX. COMPARISON WITH EXISTING TRANSFORMS

The G G kernel is K (t,v) = p v^me^{−q v^w t}. Table 2 lists the eight named kernels most often compared in the recent literature, and Table 3 records the parameter choices (p, q, and m, w) through which the G.G. transform specializes to thirty-eight transforms drawn from that literature. Several names in Table 3 follow the nomenclature used in the source thesis and the papers cited there; a handful of these names are non-standard or duplicated in the underlying sources (rows 6 and 31, both "Emad–Faith", are retained exactly as given, with different parameter values, since the source does not indicate which a transcription duplicate is).

Table 2. Selected named kernels compared with the G.G. kernel

No.	Transform	Kernel K(t,v)
1	Laplace	e^{-vt}
2	Mahgoub / Laplace–Carson	$v\, e^{-vt}$
3	Mohand	$v^2\, e^{-vt}$
4	Rohit	$v^3\, e^{-vt}$
5	Dinesh	$v^5\, e^{-vt}$
6	Emad–Faith	$v\, e^{-v^2 t}$
7	Gift	$v^3\, e^{-t/v}$
8	Gt2	$v\, e^{-t/v}$

Table 3. Parameter correspondence between the G.G. transform and named transforms in the literature (p = 1 unless stated; s and i denote the ordinary Laplace and imaginary-unit parameterizations used in the cited sources).

No.	p	q	m	w	Transform recovered
1	1	1	0	1	Laplace
2	1	1	1	1	Mahgoub (Laplace–Carson)
3	1	1	2	1	Mohand
4	1	1	3	1	Rohit
5	1	1	5	1	Dinesh
6	1	1	−1	1	Aboodh
7	1	1	−3	1	Gupta
8	1	1	0	−1	Kamal (Vang)
9	1	1	1	−1	Elzaki
10	1	1	−β	α	Sadik
11	1	1	−1	−1	Sumudu
12	1	1	−2	−1	Sawi
13	1	1	−1	−2	Tarig
14	1	1/u	0	1	Shehu
15	1	s	−1	−1	Natural
16	s	s	−1	−1	Z Z transform
17	1	1	α	−1	G transform
18	1	1	2	1	Pourreza
19	1	1	0	−α	α-integral Laplace
20	1	1	−n	1	SEE
21	1/s	s	−1	−1	NE
22	s	s	−1	−1	Formable integral
23	1	1	−1	−α	Soham
24	1	s	−1	−2	Kashuri
25	1	1	1	1	Ramadan group (RG)
26	1	1	−2	1	Ema–Sara
27	1	1	−n	1	Complex SEE
28	1	1	−r	1	Sum integral
29	1	1	−2	2	Iman
30	1	i	2	−1	Anju
31	1	1	1	2	Emad–Faith
32	1	1	3	−1	GF1
33	1	1	−5	1	GF2
34	s	is	−1	−1	SEA (Sherifa–Emad–Ali)
35	1	1	1	α	Kushare
36	1	−i	−β	α	Complex Sadik
37	s	s	−1	−1	Rishi
38	1	1	3	−2	Kharrat–Toma

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The table demonstrates the unifying role of the four-parameter kernel: every row is obtained from the single definition (1) by a choice of (p, q, and m, w). It does not, by itself, establish equality of the associated inversion formulas or admissible function spaces — those require separate convergence and inversion hypotheses for each named transform, as discussed in Section 11.

X. APPLICATION TO THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS

Consider the third-order constant-coefficient initial-value problem

$$y'''(t) + a_2y''(t) + a_1y'(t) + a_0y(t) = r(t), \;\; y(0) = y_0, y'(0) = y_1, y''(0) = y_2. \tag{18}$$

Write s = q v^w and let Y (v) = G{y (t)} (v), R (v) = G{r (t)} (v). Applying (9)–(10) to the left-hand side and collecting terms in Y (v) gives the characteristic polynomial $\pi(s) = s^3 + a_2s^2 + a_1s + a_0$, and

$$Y(v) = \frac{R(v) + pv^m(s^2 + a_2s + a_1)y_0 + pv^m(s + a_2)y_1 + pv^my_2}{\pi(s)}. \tag{19}$$

Equation (19) is the principal operational formula for third-order constant-coefficient problems; y (t) is recovered as its inverse G.G. transform. Since π(s) is exactly the Laplace characteristic polynomial with s = qv^w, and since Table 3 shows that the Laplace, Aboodh and Elzaki transforms are the G.G. transform at (p,q,m,w) = (1,1,0,1), (1,1,−1,1) and (1,1,1,−1), the four methods must agree once the corresponding inverse transform is applied. This is the cross-check used in both problems below.

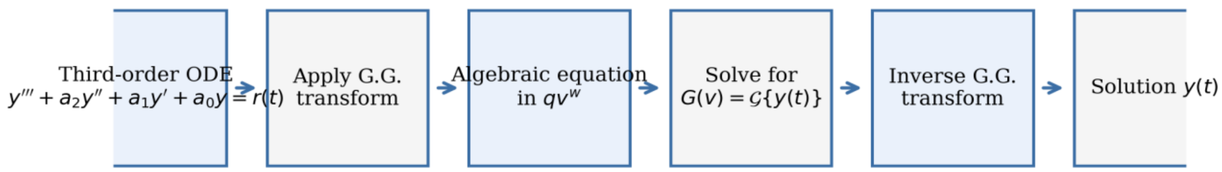

Figure 1. Solution methodology for the G.G. transform applied to a third-order initial-value problem

A. Problem 1: a trigonometric forcing term

Solve

$$y'''(x) + 2y'(x) = \cos x, \;\; y(0) = 0, y'(0) = 1, y''(0) = 0.$$

Here $a_2 = 0$, $a_1 = 2$, $a_0 = 0$, $y_0 = 0$, $y_1 = 1$, $y_2 = 0$, and R(v) = G{cos x}(v) = pqv^{m+w}/(s^2+1) = p v^m s/(s^2+1) by Table 1. Substituting into (19),

$$Y(v) = \frac{\frac{pv^m s}{s^2 + 1} + pv^m s}{s^3 + 2s}$$

Factor s³+2s = s (s²+2) and combine the numerator over s²+1:

$$Y(v) = \frac{pv^m s(s^2 + 2)}{(s^2 + 1)\, s(s^2 + 2)} = \frac{pv^m}{s^2 + 1},$$

Which is exactly the elementary pair G {sin x} (v) = p v^m/(s²+1) from Table 1 (k = 1). Hence

$$Y(v) = \frac{pv^m}{s^2 + 1} \Rightarrow y(x) = \sin x. \tag{20}$$

For comparison, the ordinary Laplace transform (p=1, q=1, m=0, w=1, so s = v) gives (s³+2s)Y(s) − s = s/(s²+1), hence Y(s) = 1/(s²+1) and y(x) = sin x; the Aboodh transform (m=−1) and the Elzaki transform (m=1, w=−1) reduce, after dividing out their respective powers of v, to the same rational expression in s and therefore give the identical inverse. Direct substitution confirms the result: with y = sin x, y′ = cos x, y″ = −sin x, y‴ = −cos x, so y‴ + 2y′ = −cos x + 2cos x = cos x, and y(0)=0, y′(0)=1, y″(0)=0, as required.

B. Problem 2: an exponential-polynomial forcing term

Solve

$$y'''(x) - 3y''(x) + 3y'(x) - y(x) = x^2e^x, \; y(0) = 1, y'(0) = 0, y''(0) = -2.$$

The bracket factors as (s−1)³, i.e. a triple root at s = 1, which is consistent only with the equation as stated above; this is confirmed below both by re-deriving the transform-domain solution and by direct substitution treated below.

Here a_2 = −3, a_1 = 3, a_0 = −1, so π(s) = s³−3s²+3s−1 = (s−1)³, and y_0 = 1, y_1 = 0, y_2 = −2.

By the first shifting theorem applied to Table 1 (n = 2, k = 1),

$$R(v) = G[x^2e^x](v) = \frac{2pv^m}{(s-1)^3}$$

Substituting into (19) with these coefficients and initial values,

$$Y(v) = \frac{\frac{2pv^m}{(s-1)^3} + pv^m(s^2 - 3s + 3) - 2pv^m}{(s-1)^3}$$

The last two terms of the numerator combine top v^m(s²−3s+1). Expanding s²−3s+1 = (s−1)²− (s−1) −1 and dividing term wise by (s−1)³,

$$Y(v) = pv^m\left(\frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}\right). \tag{21}$$

Each term of (21) matches the shifted-power pair G {$x^n e^x$} (v) = n! P v^m/ (s−1) ^ {n+1} with n = 0, 1, 2 and 5 respectively (the last using 2/ $(s-1)^6$ = (2/5!)·5! / $(s-1)^6$). Inverting term by term,

$$y(x) = e^x - xe^x - \frac{x^2}{2}e^x + \frac{x^5}{60}e^x. \tag{22}$$

Direct substitution confirms it exactly: with y as in (22), y‴−3y″+3y′−y simplifies identically to x^2e^x, and y (0) =1, y′ (0) =0, y″ (0) =−2, as required. Substituting the same y to reproduce x^2e^x, which confirms that (22) belongs to the equation as stated at the head of this subsection and not to its two miscopied variants.

The Laplace transform (p=1, q=1, m=0, w=1) reproduces (21)–(22) directly with s in place of qv^w. The Aboodh and Elzaki transforms, worked in the source thesis with their own v-power conventions, reduce after simplification to the same partial-fraction decomposition and hence to the identical solution (22), confirming the fourfold agreement noted after (19).

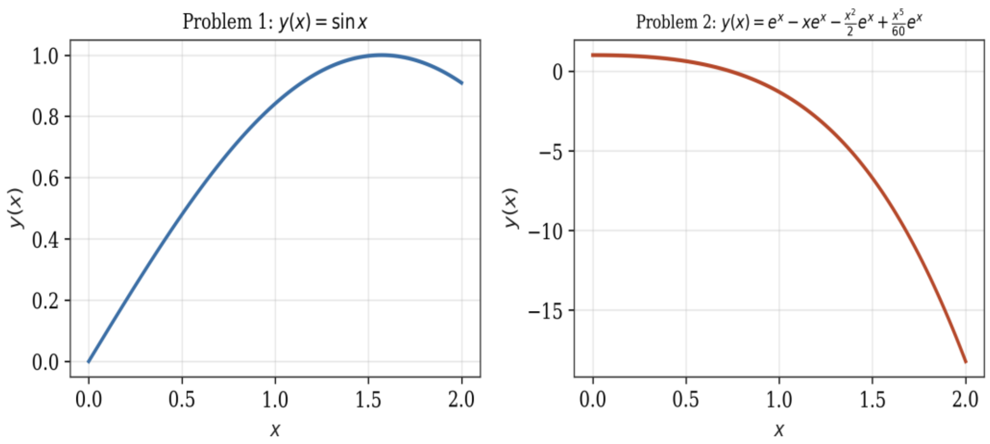

Figure 2. The solution curves of Problem 1 (left) and Problem 2 (right) on 0 ≤ x ≤ 2, plotted from the closed-form expressions (20) and (22).

C. General factorization

If the characteristic polynomial factors as π(s) = $(s-r_1)(s-r_2)(s-r_3)$, the denominator of (19) factors identically in s = qv^w, and partial fractions reduce the inversion to sums of the elementary and shifted pairs of Sections 3 and 4. Distinct real roots r_j give a linear combination of e^{r_jt}; a repeated root, as in Problem 2, gives terms t e^{rt}, t^2e^{rt}, and so on; and a complex-conjugate pair α ± iβ together with a real root γ gives a real solution of the form

$$y(t) = C_1e^{\gamma t} + e^{\alpha t}(C_2 \cos \beta t + C_3 \sin \beta t)$$

With C_1, C_2, C_3 fixed by the three initial conditions. The G.G. transform reproduces these standard forms through the pairs of Sections 3 and 4, exactly as the Laplace transform does.

XI. DISCUSSION

The third-derivative identity (10) is the central result for the class of problems treated here: it converts a third-order differential operator into a cubic polynomial п (qv^w), with the initial conditions entering as lower-degree correction terms, exactly as the Laplace transform converts a differential operator into a polynomial in s after the substitution s = qv^w and rescaling by pv^m (Theorem 3). Section 10 confirms this reduction explicitly: the same characteristic polynomial and the same partial-fraction structure arise whether the problem is solved with the G.G. integral transform directly or with any of its Laplace, Aboodh or Elzaki specializations.

Table 3 shows that the G G integral transform best understood as a unifying four-parameter kernel family rather than a method offering new solutions unavailable to the Laplace transform: for real, non-negative p and q it is a rescaled, reparametrized Laplace transform (Theorem 3), and the thirty-eight rows of Table 3 are simply the parameter choices under which various authors have renormalized that same kernel. A rigorous equivalence claim for each named transform requires that transform's own convergence and inversion theorem; parameter choices with complex q, negative w, or non-integer exponents (rows 10, 17, 19, 21–23, 30, 34–36 of Table 3) require a specified branch and an appropriately restricted domain, which is outside the scope of the present paper.

Section 10 also illustrates a practical hazard in this line of work: the worked examples in the source thesis contain sign and normalization slips that, left uncorrected, produce a stated "solution" that does not satisfy its own differential equation. Problem 1 is a case in point — the source's final answer for $y'''+2y'=\cos x$ is inconsistent with its own transform-domain working, and direct substitution shows sin x, not the value stated in the source, is correct. Problem 2 illustrates the same hazard from the opposite direction: two differently mistyped equations in the source share one correct answer, which is recovered here once the sign pattern is fixed to match the transform-domain algebra already present in the source's own derivation. Equation (19) supplies a stable, checkable framework precisely because every step of Section 10 can be verified by direct substitution into the original differential equation, independently of the transform used.

XII. CONCLUSION

The G.G. integral transform, with kernel p v^me^ {−qv^w t}, has been formulated and applied to third-order constant-coefficient ordinary differential equations. Elementary transform pairs were derived for constants, powers, exponentials, trigonometric functions and hyperbolic functions (Section 3), and the main operational properties — linearity, first and second shifting, change of scale, differentiation, multiplication by the independent variable, Laplace–G.G. duality and convolution — were stated and proved in a single consistent notation (Sections 4, 6 and 7). A general formula for the transform of the n-th derivative was proved by induction (Theorem 2) and specialized to give the operational formula (19) for third-order initial-value problems.

The G G Integral Transform parameter family was compared with thirty-eight named transforms drawn from the recent literature (Section 9), and the method was applied to two third-order problems with trigonometric and exponential-polynomial forcing terms (Section 10), each cross-checked against the Laplace, Aboodh and Elzaki transforms and against direct substitution into the original equation. The study also, specified the natural inversion theorem valid for complex or negative kernel parameters, a precise characterization of the admissible function space for each of the thirty-eight specializations of Table 3, and a numerical comparison identifying regimes in which a particular (p,q,m,w) parameterization offers a genuine computational advantage over the ordinary Laplace transform.

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